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A Study on Two Types of Improper Fractional Integrals

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Abstract: **In this paper, the concept of fractional analytic function and a new multiplication of fractional analytic functions play important roles. We use Jumarie type of modified Riemann-Liouville (R-L) fractional derivative to study two types of improper fractional integrals.**

Keywords: **fractional analytic function, new multiplication, Jumarie type of modified R-L fractional derivative, improper fractional integrals.**

I. INTRODUCTION

Fractional calculus includes the derivative and integral of any real order or complex order. In recent years, fractional calculus has achieved considerable popularity and attention, due to its application in various fields such as elasticity, mechanics, dynamics, electronics, modelling, physics, mathematical economics, and control theory [1-11]. Fractional calculus is different from traditional calculus. There is no unique definition of fractional derivative and integral. Commonly used definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald letinikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [4, 12, 23]. On the other hand, the application of fractional calculus in fractional differential equations can be referred to [13-22].

In this article, we solve the following two types of improper fractional integrals:

$$\left({}_0I^{\alpha}_{+\infty}\right)[E_\alpha(\lambda x^\alpha)\otimes cos_\alpha(\mu x^\alpha)]\,, \qquad (1)$$

$$\left({}_0I^{\alpha}_{+\infty}\right)[E_\alpha(\lambda x^\alpha)\otimes sin_\alpha(\mu x^\alpha)]. \qquad (2)$$

Where $0<\alpha\le 1$, and λ,μ are real numbers with $\lambda^2+\mu^2\neq 0$. The fractional analytic functions such as fractional exponential function, fractional sine and cosine functions are discussed. The new multiplication $\otimes$ of fractional analytic functions is a natural generalization of ordinary multiplication in calculus. And Jumarie′s modified Riemann-Liouville fractional derivative is used to study the above two improper fractional integrals.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1: Suppose that α is a real number, and n is a positive integer. The Jumarie type of modified Riemann-Liouville fractional derivative [12] is defined as

$$\left({}_{x_0}D^{\alpha}_{x}\right)[f(x)]=\begin{cases}\frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}(x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}(x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0\le\alpha<1\\ \frac{d^n}{dx^n}\left({}_{x_0}D^{\alpha-n}_{x}\right)[f(x)], & \text{if } n\le\alpha<n+1\end{cases} \qquad (3)$$

where $\Gamma(u)=\int_0^\infty t^{u-1}e^{-t}dt$ is the gamma function defined on $u>0$. In addition, we define the α-fractional integral of $f(x)$ by $\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\left({}_{x_0}D_x^{-\alpha}\right)[f(x)]$, where $\alpha>0$. If $\left({}_{x_0}I_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called an α-fractional integrable function. We have the following properties [23].

Proposition 2.2: *If α,β,c are real numbers and $\beta\geq\alpha>0$, then*

$$ {}_0D_x^{\alpha}\left[x^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \quad (4)$$

and

$$ {}_0D_x^{\alpha}[c]=0. \quad (5)$$

In the following, we define the fractional analytic function.

Definition 2.3 ([25])**:** Assume that x,x_0 and a_n are real numbers, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval (x_0-s,x_0+s), then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0, where s is the radius of convergence about x_0. In addition, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

Next, some fractional analytic functions are introduced.

Definition 2.4 ([24])**:** The Mittag-Leffler function is defined as

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \quad (6)$$

where α is a real number, $\alpha\geq 0$, and z is a complex number.

Definition 2.5 ([20])**:** Let $0<\alpha\leq 1$, and λ,x be real numbers. $E_\alpha(\lambda x^\alpha)=\sum_{k=0}^{\infty}\frac{\lambda^k x^{k\alpha}}{\Gamma(k\alpha+1)}$ is called α-fractional exponential function and the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(\lambda x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \quad (7)$$

and

$$sin_\alpha(\lambda x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k+1}x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \quad (8)$$

Remark 2.6: If $\alpha=1$, $\lambda=1$, then $cos_1(x)=cosx$, and $sin_1(x)=sinx$.

Notation 2.7: Suppose that $z=a+ib$ is a complex number, where $i=\sqrt{-1}$, and a,b are real numbers. Then a, the real part of z, is denoted as $\text{Re}(z)$; b, the imaginary part of z, is denoted as $\text{Im}(z)$.

Proposition 2.8 (fractional Euler's formula): *Assume that* $0<\alpha\leq 1$, *then*

$$E_\alpha(ix^\alpha)=cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha)\,. \quad (9)$$

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.9 ([25])**:** Assume that $0<\alpha\leq 1$, $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}x^{k\alpha}, \quad (10)$$

$$g_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\alpha+1)}x^{k\alpha}. \quad (11)$$

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Then we define

$$f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m \right) x^{k\alpha}. \quad (12)$$

Definition 2.10: Let $f_\alpha(\lambda x^\alpha)$, $g_\alpha(\lambda x^\alpha)$ be two α-fractional analytic functions. If $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\lambda x^\alpha) = 1$, then $g_\alpha(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\lambda x^\alpha)$, and is denoted as $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes -1}$.

Remark 2.11: We note that the $\otimes$ multiplication satisfies the commutative law and the associate law, and it is a generalization of ordinary multiplication, since the $\otimes$ multiplication becomes the traditional multiplication if $\alpha = 1$.

Proposition 2.12 ([20])**:** *Let* $0 < \alpha \leq 1$, *and* λ *be a real number, then*

$$E_\alpha(\lambda x^\alpha) \otimes E_\alpha(-\lambda x^\alpha) = 1. \quad (13)$$

III. MAJOR RESULTS

The followings are major results in this article.

Theorem 3.1: *Let* $0 < \alpha \leq 1$, *and* λ, μ *be real numbers such that* $\lambda^2 + \mu^2 \neq 0$. *Then*

$$\left({}_0I_x^\alpha\right)[E_\alpha(\lambda x^\alpha) \otimes cos_\alpha(\mu x^\alpha)] = E_\alpha(\lambda x^\alpha) \otimes \frac{\lambda cos_\alpha(\mu x^\alpha) + \mu sin_\alpha(\mu x^\alpha)}{\lambda^2+\mu^2} - \frac{\lambda}{\lambda^2+\mu^2}, \quad (14)$$

and

$$\left({}_0I_x^\alpha\right)[E_\alpha(\lambda x^\alpha) \otimes sin_\alpha(\mu x^\alpha)] = E_\alpha(\lambda x^\alpha) \otimes \frac{\lambda sin_\alpha(\mu x^\alpha) - \mu cos_\alpha(\mu x^\alpha)}{\lambda^2+\mu^2} + \frac{\mu}{\lambda^2+\mu^2}. \quad (15)$$

Proof $\quad \left({}_0I_x^\alpha\right)[E_\alpha(\lambda x^\alpha) \otimes cos_\alpha(\mu x^\alpha)]$

$$= \left({}_0I_x^\alpha\right)\left[\mathrm{Re}[E_\alpha((\lambda + i\mu)x^\alpha)]\right]$$

$$= \mathrm{Re}\left[\left({}_0I_x^\alpha\right)[E_\alpha((\lambda + i\mu)x^\alpha)]\right] - \frac{\lambda}{\lambda^2+\mu^2}$$

$$= \mathrm{Re}\left[\frac{1}{\lambda+i\mu} E_\alpha((\lambda + i\mu)x^\alpha\right] - \frac{\lambda}{\lambda^2+\mu^2}$$

$$= \mathrm{Re}\left[\frac{\lambda - i\mu}{\lambda^2 + \mu^2}[E_\alpha(\lambda x^\alpha) \otimes cos_\alpha(\mu x^\alpha) + iE_\alpha(\lambda x^\alpha) \otimes sin_\alpha(\mu x^\alpha)]\right] - \frac{\lambda}{\lambda^2 + \mu^2}$$

$$= E_\alpha(\lambda x^\alpha) \otimes \frac{\lambda cos_\alpha(\mu x^\alpha) + \mu sin_\alpha(\mu x^\alpha)}{\lambda^2+\mu^2} - \frac{\lambda}{\lambda^2+\mu^2}.$$

On the other hand,

$$\left({}_0I_x^\alpha\right)[E_\alpha(\lambda x^\alpha) \otimes sin_\alpha(\mu x^\alpha)]$$

$$= \left({}_0I_x^\alpha\right)\left[\mathrm{Im}[E_\alpha((\lambda + i\mu)x^\alpha)]\right]$$

$$= \mathrm{Im}\left[\left({}_0I_x^\alpha\right)[E_\alpha((\lambda + i\mu)x^\alpha)]\right] + \frac{\mu}{\lambda^2+\mu^2}$$

$$= \mathrm{Im}\left[\frac{1}{\lambda+i\mu} E_\alpha((\lambda + i\mu)x^\alpha\right] + \frac{\mu}{\lambda^2+\mu^2}$$

$$= \mathrm{Im}\left[\frac{\lambda-i\mu}{\lambda^2+\mu^2}[E_\alpha(\lambda x^\alpha) \otimes cos_\alpha(\mu x^\alpha) + iE_\alpha(\lambda x^\alpha) \otimes sin_\alpha(\mu x^\alpha)]\right] + \frac{\mu}{\lambda^2+\mu^2}$$

$$= E_\alpha(\lambda x^\alpha) \otimes \frac{\lambda sin_\alpha(\mu x^\alpha) - \mu cos_\alpha(\mu x^\alpha)}{\lambda^2+\mu^2} + \frac{\mu}{\lambda^2+\mu^2}.$$

Q.e.d.

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Lemma 3.2: *Suppose that* $0 < \alpha \leq 1$ *and* $\lambda < 0$. *Then*

$$\lim_{x\to+\infty} E_\alpha(\lambda x^\alpha) = 0. \tag{16}$$

Proof Since $E_\alpha(\lambda x^\alpha) \otimes E_\alpha(-\lambda x^\alpha) = 1$, it follows that

$$\lim_{x\to+\infty} E_\alpha(\lambda x^\alpha) \otimes \lim_{x\to+\infty} E_\alpha(-\lambda x^\alpha) = \lim_{x\to+\infty} E_\alpha(\lambda x^\alpha) \otimes E_\alpha(-\lambda x^\alpha) = 1\,. \tag{17}$$

By $\lim_{x\to+\infty} E_\alpha(-\lambda x^\alpha) = \infty$, we have $\lim_{x\to+\infty} E_\alpha(\lambda x^\alpha) = 0$.

Q.e.d.

Theorem 3.3: *Assume that* $0 < \alpha \leq 1$, *and* λ, μ *are real numbers with* $\lambda < 0$. *Then the improper* α*-fractional integrals*

$$\left({}_0I^\alpha_{+\infty}\right)[E_\alpha(\lambda x^\alpha) \otimes cos_\alpha(\mu x^\alpha)] = -\frac{\lambda}{\lambda^2+\mu^2}\,, \tag{18}$$

and

$$\left({}_0I^\alpha_{+\infty}\right)[E_\alpha(\lambda x^\alpha) \otimes sin_\alpha(\mu x^\alpha)] = \frac{\mu}{\lambda^2+\mu^2}\,. \tag{19}$$

Proof Since $\lambda < 0$, using Theorem 3.1 and Lemma 3.2 yields

$$\begin{aligned} &\left({}_0I^\alpha_{+\infty}\right)[E_\alpha(\lambda x^\alpha) \otimes cos_\alpha(\mu x^\alpha)] \\ &= \lim_{x\to+\infty}\left({}_0I^\alpha_x\right)[E_\alpha(\lambda x^\alpha) \otimes cos_\alpha(\mu x^\alpha)] \\ &= \lim_{x\to+\infty}\left(E_\alpha(\lambda x^\alpha) \otimes \frac{\lambda cos_\alpha(\mu x^\alpha) + \mu sin_\alpha(\mu x^\alpha)}{\lambda^2+\mu^2} - \frac{\lambda}{\lambda^2+\mu^2}\right) \\ &= -\frac{\lambda}{\lambda^2+\mu^2}. \end{aligned}$$

And

$$\begin{aligned} &\left({}_0I^\alpha_{+\infty}\right)[E_\alpha(\lambda x^\alpha) \otimes sin_\alpha(\mu x^\alpha)] \\ &= \lim_{x\to+\infty}\left({}_0I^\alpha_x\right)[E_\alpha(\lambda x^\alpha) \otimes sin_\alpha(\mu x^\alpha)] \\ &= \lim_{x\to+\infty}\left(E_\alpha(\lambda x^\alpha) \otimes \frac{\lambda sin_\alpha(\mu x^\alpha) - \mu cos_\alpha(\mu x^\alpha)}{\lambda^2+\mu^2} + \frac{\mu}{\lambda^2+\mu^2}\right) \end{aligned}$$

$$= \frac{\mu}{\lambda^2+\mu^2}.$$

Q.e.d.

IV. CONCLUSION

As mentioned above, the purpose of this paper is to solve two improper fractional integrals. In fact, these two types of improper fractional integrals are generalizations of improper integrals in classical calculus. The method we used in this paper is also similar to that in calculus. On the other hand, the new multiplication we defined plays an important role in this article, and it is a natural operation in fractional calculus. In the future, we will also use Jumarie′s modified R-L fractional derivative and the new multiplication to study the problems in engineering mathematics and fractional calculus.

REFERENCES

[1] N. Sebaa, Z. E. A. Fellah, W. Lauriks, C. Depollier,"Application of fractional calculus to ultrasonic wave propagation in human cancellous bone,"Signal Processing, vol. 86, no. 10, pp. 2668-2677, 2006.

[2] R. L. Magin,"Modeling the cardiac tissue electrode interface using fractional calculus,"Journal of Vibration and Control, vol. 14, no. 9-10, pp. 1431-1442, 2008.

[3] E. Soczkiewicz,"Application of fractional calculus in the theory of viscoelasticity,"Molecular and Quantum Acoustics, vol.23, pp. 397-404. 2002.

[4] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[5] F. Mainardi,"Fractional calculus: some basic problems in continuum and statistical mechanics,"Fractals and Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi, Eds., pp. 291-348, Springer, Wien, Germany, 1997.

[6] Mohd. Farman Ali, Manoj Sharma, Renu Jain,"An application of fractional calculus in electrical engineering,"Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[7] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[8] R. L. Bagley and P. J. Torvik,"A theoretical basis for the application of fractional calculus to viscoelasticity,"Journal of Rheology, vol. 27, no. 3, pp. 201-210, 1983.

[9] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, INC. 1974.

[10] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[11] R. Almeida, N. R. Bastos, and M. T. T. Monteiro,"Modeling some real phenomena by fractional differential equations," Mathematical Methods in the Applied Sciences, vol. 39, no. 16, pp. 4846-4855, 2016.

[12] G. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results," Computers and Mathematics with Applications, vol. 51, pp.1367-1376, 2006.

[13] C. -H. Yu, "Fractional Clairaut's differential equation and its application,"International Journal of Computer Science and Information Technology Research, vol. 8, no. 4, pp. 46-49, 2020.

[14] C. -H. Yu, "Separable fractional differential equations, "International Journal of Mathematics and Physical Sciences Research, vol. 8, no. 2, pp. 30-34, 2020.

[15] C. -H. Yu,"Integral form of particular solution of nonhomogeneous linear fractional differential equation with constant coefficients, "International Journal of Novel Research in Engineering and Science, vol. 7, no. 2, pp. 1-9, 2020.

[16] C. -H. Yu, "A study of exact fractional differential equations,"International Journal of Interdisciplinary Research and Innovations, vol. 8, no. 4, pp. 100-105, 2020.

[17] C. -H. Yu, "Research on first order linear fractional differential equations,"International Journal of Engineering Research and Reviews, vol. 8, no. 4, pp. 33-37, 2020.

[18] C. -H. Yu, "Method for solving fractional Bernoulli's differential equation, "International Journal of Science and Research, vol. 9, no. 11, pp. 1684-1686, 2020.

[19] C. -H. Yu, "Using integrating factor method to solve some types of fractional differential equations,"World Journal of Innovative Research, vol. 9, no. 5, pp. 161-164, 2020.

[20] C. -H. Yu, "Differential properties of fractional functions,"International Journal of Novel Research in Interdisciplinary Studies, vol. 7, no. 5, pp. 1-14, 2020.

[21] C. -H. Yu, "Study on fractional Newton's law of cooling," International Journal of Mechanical and Industrial Technology, vol. 9, no. 1, pp. 1-6, 2021,

[22] C. -H. Yu, "A new insight into fractional logistic equation," International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021,

[23] U. Ghosh, S. Sengupta, S. Sarkar, and S. Das,"Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, "American Journal of Mathematical Analysis, vol. 3, no. 2, pp.32-38, 2015.

[24] J. C. Prajapati, "Certain properties of Mittag-Leffler function with argument x^{α}, $\alpha > 0$, "Italian Journal of Pure and Applied Mathematics, vol. 30, pp. 411-416, 2013.

[25] C. -H. Yu, "Study of fractional analytic functions and local fractional calculus," International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.